Sorting Analysis
Average Case Insertion Sort

Average number of comparisons to insert a value into a sorted array containing n values:

\[
\frac{1}{n+1} \sum_{i=1}^{n+1} i = \frac{1}{n+1} \left( \frac{(n+1)(n+2)}{2} \right) = \frac{n+2}{2}
\]
Average number of comparisons to sort an array of $n$ values using an insertion sort:

\[
\frac{n-1}{2} \cdot \frac{c + 2}{2} = \frac{1}{2} \sum_{c=1}^{n-1} (c+2) = \frac{1}{2} \left( \sum_{c=1}^{n-1} c + \sum_{c=1}^{n-1} 2 \right) = \frac{1}{2} \left( \frac{n(n-1)}{2} + 2(n-1) \right) = \frac{n(n-1)}{4} + (n-1)
\]
Merge Sort Analysis (n is a power of 2)

Recurrence Relation for Merge Sort

\[ T(n) = 2T(n/2) + n \]

How to solve?

Restructure the equation so terms cancel

Divide by sides by \( n \)

\[ \frac{T(n)}{n} = \frac{2T(n/2) + 1}{n} \]

\[ \frac{T(n)}{n} = \frac{T(n/2)}{n/2} + 1 \]
Merge Sort Analysis (n is a power of 2)

Let's assume \( n \) is a power of 2.

\[
\begin{align*}
T(n) &= T(n/2) + 1 \\
T(n/2) &= T(n/4) + 1 \\
T(n/4) &= T(n/8) + 1 \\
&\vdots \\
T(2) &= T(1) + 1
\end{align*}
\]
Merge Sort Analysis (n is a power of 2)

Set the sum of the LHS equal to the sum of the RHS and cancel common terms.

Since $n$ is a power of 2 there are $\log_2 n$ equations and thus $\log_2 n$ of 1s

$$\frac{T(n)}{n} = T(1) + \log_2 n$$

$$T(n) = nT(1) + n\log_2 n = n + n\log_2 n$$
QuickSort Analysis

Recurrence relation

\[ T(0) = T(1) = 1 \]

\[ T(N) = T(i) + T(N-i-1) + N, \text{ for } N > 1 \]
QuickSort Analysis

Worst Case Analysis

\[ T(N) = T(N-1) + N, \text{ for } N > 1 \]

Telescope the equation

\[ T(N) = T(N-1) + N \]
\[ T(N-1) = T(N-2) + (N-1) \]
\[ T(N-2) = T(N-3) + (N-2) \]
\[ \vdots \]
\[ T(2) = T(1) + 2 \]

Sum the equations and cancel matching terms in the left and right resulting

\[ T(N) = T(1) + N + (N-1) + \ldots + 2 \]
QuickSort Analysis

Best Case

\[ T(N) = 2T(N/2) + N, \text{ for } N > 1 \]

This can be solved the same way we solved the recurrence relation for merge sort.
QuickSort Analysis

Average case

Average the recurrence equation for all possible values of \( l \) (all possible locations of the pivot element)

\[
\frac{1}{N} \sum (T(i) + T(N-i-1)) + N
\]

See textbook for a solution