Aggregative combinatorics: An introduction

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Abstract

This paper is an exploratory introduction to issues surrounding the combinatorial expressivity and computational complexity of weakly aggregative modal logic (WAL) – the latter insofar as it relates to the former, but also more generally. We begin by showing how WAL allows us to define a natural hierarchy of (recognition) optimization problems not generally available in logics as strong as \( K \). In this respect we illustrate that an important advantage of the Aggregative Programme in modal logic (as dubbed by its progenitors, Jennings and Schotch) over the standard modal paradigm, which views \( K \) as the weakest normal modal logic, has to do with its expressive, discriminatory power. As an extended example in this context we formulate the Four Colour Theorem (Every planar graph is 4-colourable) (4CT) as a rule schema in WAL which preserves validity with respect to quinary relational frames if and only if 4CT is true. Accordingly, because WAL is decidable, additional motivations for exploring its combinatorial expressivity arise from issues of decidability in graph theory. In closing we analyze the complexity of the satisfiability problem for these logics.

1 Introduction

Whereas the standard (Kripkean) approach to modal logic views \( K \) as the base of the lattice of inclusion according to which all normal modal logics are ordered (Schotch and Jennings, 1980), on the Aggregative Programme in modal logic, this paradigm is inverted to accommodate the study of logics weaker than \( K \) (Jennings and Schotch, 1981). First espoused in detail in (Schotch and Jennings, 1980), motivations for this programme have included (a) its ability to discern semantical limitations of the standard approach, particularly with respect to first-order (in)definability in modal logic,\(^1\) and the fact that, (b) like the logics \( K \) and \( E \), determined by the class of Scott-Montague, or neighborhood, and binary relational frames, respectively, the class of \( K_n \) logics \( (n \in \mathbb{Z}^+) \), with which aggregativity theory is mainly concerned, is also determined by an unrestricted class of structures (Jennings and Schotch, 1981). Specifically, weakly aggregative logics, as they are also called, are determined by the class of \((n+1)\)-ary relational frames, their alias borrowed from a natural combinatorial property modelled by these structures, given the truth-condition defined for the necessity operator.

We begin with a survey of the basic ideas and results associated with these logics, before proceeding to our new results.

Definition 1.1. Let \( U \) be a non-empty set, and let \( R \subseteq U^{n+1} \). Then \( \mathfrak{F} = (U, R) \) is an \((n+1)\)-ary relational frame (‘frame’, for short). Further, if \( \mathfrak{F} \) is a frame and \( V \) is a valuation function from propositional variables to subsets of \( U \) which is defined as usual for Boolean operators and extended to modal formulae via:

\[
V(\Box A) := \{ x \in U \mid \forall (y_1, ..., y_n) \in U^n, Rxy_1, ..., y_n \Rightarrow \exists i (1 \leq i \leq n) : y_i \in V(A) \},
\]

then the pair \((\mathfrak{F}, V)\) is an \((n+1)\)-ary relational model \( \mathfrak{M} \) (on \( \mathfrak{F} \)).

\(^1\)For instance, Jennings et al. have shown that neither \( [D] : \square p \rightarrow \diamond p \) nor \( [C] : \diamond \square p \rightarrow \square \diamond p \) is first-order definable using only a single \( n \)-ary predicate \((n \geq 3)\) (Jennings, Schotch, and Johnston, 1980), (Jennings, Schotch, and Johnston, 1981), (Schotch, Jennings, and Johnston, ), notwithstanding that both formulae are first-order definable using a single binary predicate.
In proof-theoretic terms, this combinatorial property can be seen as the condition that for each \( n > 1 \), any complete axiomatization of the logic determined by the class of frames is distinguished from a complete axiomatization of its predecessor by a *weakening* of the extent to which individually necessary formulae are jointly necessary. In the system \( K_n \) this distinguishing condition takes the following form:

\[
[K_n] : \vdash \Box p_1 \land \ldots \land \Box p_{n+1} \rightarrow \bigvee_{i,j \in [n+1] \mid (i \neq j)} p_i \land p_j,
\]

which, enjoined with:

\[
[RR] : \vdash A \rightarrow B \Rightarrow \vdash \Box A \rightarrow \Box B,
\]

\[
[RN] : \vdash A \Rightarrow \vdash \Box A,
\]

\[
[RPL] : \vdash_{PL} A \Rightarrow \vdash A,
\]

constitutes a sound and complete axiomatization of the logics of \((n+1)\)-ary frames (Apostoli and Brown, 1995), (Nicholson, Jennings, and Sarenac, 2000), (Urquhart, 1995).\(^2\) Thus, using the idiom of aggregativity, the system \( N \), consisting solely of \([RR],[RN]\), and \([RPL]\) (see (Jennings and Schotch, 1981)), can be said to be non-aggregative, and the system \( K (= K_1) \), maximally aggregative. In fact, it is proved in (Jennings and Schotch, 1981) that the system \( N \) is the intersection of the denumerable sequence of \( K_n \) systems, of which the limit is, to use the standard idiom, the ‘minimal’ logic, \( K \). Note that these logics, while weak with respect to aggregation, are in other respects typical modal logics; that is, they are normal, due to the presence of \([RN]\), and the relation \( \vdash \) has all the properties, such as monotonicity and transitivity, usually associated with a consequence-relation.

In this paper, we consider yet another limitation of the standard modal paradigm, viz., that it obfuscates natural connections between relational modal semantics and graph colouring problems. For example, the perspective which views \( K \) as the weakest logic fails to see that the validity of \([K] (= [K_1])\) (on binary frames) is equivalent to the 1-uncolourability of the complete graph on two vertices, \( K_2 \).\(^3\) Similarly occluded is the fact that completeness for \( K \) is essentially a consequence of the sufficiency of binary conjunction for the logical formulation of any 1-uncolourable graph (Nicholson, Jennings, and Sarenac, 2000). More generally it is possible to show that for each \( n \geq 1 \),

1. the soundness of \( K_n \) is equivalent to the \( n \)-uncolourability of \( K_{n+1} \), and
2. the completeness of \( K_n \) is equivalent to the functional completeness of \([RPL]\) enjoined with the truth function \( \frac{1}{2} \bigvee_{i,j \in [n+1] \mid (i \neq j)} p_i \land p_j \), with respect to the derivation of all (uncolouring) formulations of \( n \)-uncolourable graphs (Nicholson, Jennings, and Sarenac, 2000), (Nicholson, Jennings, and Sarenac, 2001).

We don’t go into details regarding these specific facts here; the reader is directed to the associated references. Instead, we focus on the following new results:

I. the ability of \( K_n \) to represent problems from various branches of combinatorics, and mathematics more generally – specifically, those in which the Four Colour Theorem finds equivalent expression (see (Saaty, 1972) for an extensive list), and

II. the expressivity of \( K_n \) with respect to structural relationships between distinct combinatorial problems – specifically, optimization problems involving graph colourings.

In the epilogue we show that:

III. the satisfiability problem for \( K_n \) is PSPACE-complete in general, and NP-complete for formulae of bounded modal operator depth.

The paper is structured as follows: (II) is presented in Section 2, and (I) in Section 3; Sections 4 and 5 address (III) and a related conjecture of Vardi (Vardi, 1989).

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\(^2\)A dual axiomatization can be found in (Nicholson, Jennings, and Sarenac, 2001).

\(^3\)We use the notation ‘\( K_m \)’ to denote the isomorphism class of complete graphs on \( m \) vertices (\( m > 0 \)). Since it is standard practice among graph-theorists to identify this class with any of its elements, we may also speak of the complete graph \( K_m \) on \( m \) vertices, without breaking with convention.
2 The Colouring Fragment

One reason for interest in weakly aggregative modal logics is that they allow us to define a natural class of combinatorial problems not generally available in logics as strong as $K$. Essentially this is because the language of aggregativity can be equivalently recast in the language of optimization problems whose solutions involve a determination of (un)colourability properties of graphs. Inasmuch as an optimization problem $P$ can be seen as a reasoning problem whose domain is the solution space of an instance of $P$ (Wong, 2003), differing degrees of aggregativity can therefore be seen as determining differing degrees of expressive power with respect to the representation of a class of spatial reasoning problems. To illustrate this fact, we show how $K_n$ distinguishes between the elements of a denumerable class of recognition tasks.

For our purposes it is sufficient to understand a graph $G$ as a pair $(V, E)$ where the node or vertex set $V$ of $G$ (or $VG$) is a finite non-empty set, and the edge set $E$ (or $EG$) is a finite non-empty set of pairs $e_i = \{v, w\}$, $(v \neq w)$ such that $\bigcup_{i=1}^{[E]} e_i = V$. A graph $G$ is $m$-uncolourable ($m \in \mathbb{Z}^+$) if there is no partition of $VG$ into $m$ pairwise disjoint, jointly exhaustive, cells such that every edge of $G$ has a non-empty intersection with at least two cells; otherwise, $G$ is $m$-colourable, and any such partition is an $m$-colouring of $G$.

In order to show how the aggregativity of the $K_n$ logics is related to graph colouring, we need some way of treating graphs as modal sentences. To this end we use a standard, simple modal language $L$ that has a denumerable set $\phi = \{p_1, p_2, \ldots, p_i, \ldots\}$ of propositional variables, where sub(super)scripts may be added as required. Letting the letter $p$ range over $\phi$, a sentence (formula) $A$ of the language $L$ is defined:

$$A := p \mid \bot \mid \neg B \mid B \lor C \mid \Box B,$$

where $\rightarrow$, $\land$, $\lor$, etc. acquire their usual abbreviational status, and $\Phi$ is the set of all such formulae. Then where $G = ([j], E)$ is a graph such that $[j] \subseteq \phi$, the formulation of $G$, $F(G)$ is defined:

$$F(G) := \bigvee_{e \in E} \bigwedge_{f, h \in e} f \land h,$$

and the uncolouring formulation of $G$, $UNCOL(G)$, is set:

$$UNCOL(G) := \Box 1 \land \ldots \land \Box j \rightarrow \Box F(G).$$

Introducing the convention that for any sentence $A$ and positive integer $n$, $A$ is $(n + 1)$-valid iff for every model $\mathfrak{M}$ on any $(n + 1)$-ary frame $\mathfrak{F} = (U, R)$, $\forall x \in U, x \in V(A)$, it follows that:

**Theorem 2.1.** $\forall n \geq 1, G = (V, E)$, UNCOL$(G)$ is $(n + 1)$-valid $\iff$ $G$ is $n$-uncolourable.

We can then define the set $COLR \subseteq \Phi$:

$$COLR := \{A \mid \exists G : A = \neg UNCOL(G)\}.$$

Clearly, any formula $A \in COLR$ is true at a point in a model, i.e., is $K_n$-satisfiable, iff the corresponding graph $G$ is $n$-colourable. Further, for each $n \geq 1$, it follows that the set:

$$COLR_{K_n} = \{A \mid A \in COLR \land A \text{ is } K_n\text{-satisfiable}\}$$

is a proper subset of the same set defined for logic $K_{n+1}$. That is, we have:

**Proposition 2.2.** $\forall n, COLR_{K_n} \subset COLR_{K_{n+1}}$.

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4Accordingly it is sometimes useful to identify a graph $G$ with its edge set $E$, essentially treating $G$ as a set of pairs. Note also that edges are unordered sets of distinct elements. Thus, our graphs are undirected, and contain no self-loops. We treat edges as sets in order to keep our definition a general as possible; we are working in the context of the theory of hypergraphs, where edge-sets can consist of tuples larger than pairs. This is also why we say “at least two cells” rather than “exactly two cells” in the next sentence; in a hypergraph, a proper colouring has to colour at least two elements of each edge-set differently. See (Nicholson, Jennings, and Sarenac, 2001) for details.

5We will often use the index of a variable as an abbreviation for it (e.g., $p_i$ is abbreviated ‘$i$’), and more generally, if $V$ is a set with $j$ elements, we sometimes use the notation ‘$[j]$’ ($=\{1, 2, \ldots, j\}$) to refer to $V$, so that $[j]$ is intended to represent $|V|$.
In this way $K_n$ is able to preserve distinctions relative to a denumerable class of optimization problems – distinctions, moreover, to which $K$ is blind. To illustrate, let \textit{CHRMM NMBR} be the optimization problem which has as an instance $I_G$, for any graph $G$, the pair $(F, c)$, where $F$, the set of feasible solutions for $I_G$, is the set of $n$-colourings of $G$ ($n \in \mathbb{Z}^+$), and $c : F \to \mathbb{R}$ is a cost function satisfying the condition that $\forall f \in F, c(f) = |f|$: that is, the cost of a colouring is simply the number of colours used.\footnote{Note that it would be sufficient to define cost-function $c$ as operating from $F$ to $\mathbb{Z}^+$, since this is the range of values for the the number of colours $n$. We retain the mapping to the reals to cohere to the definition of a cost function in the general definition of an optimization problem.} Then our definition of a graph entails that for any graph $G$, there is no feasible solution $f$ of cost $c(f) \leq 1$ for instance $I_G$.\footnote{If we allow graphs to have empty edge sets then this is false, since any such graph is 1-colourable. We have elected to adopt the definition which forbids empty edge sets for illustrative purposes in this context, and notational convenience elsewhere. It is easily checked that no significant difference results with respect to our main theorem (2.3) in this section, since on both definitions the decision task corresponding to $\text{COLR}_{K_n}$ is more or less trivially a member of complexity class $P$.} Whence, in fact, that $K(= K_1)$ is maximally aggregative; equivalently, the solution space $F$ of the recognition version “Is there $f \in F$ such that $c(f) \leq r$?” of \textit{CHRMM NMBR}, when $r$ is restricted to 1, is empty. But, plainly, the $K$-unsatisfiability of a formula $A$ does not imply its $K_n$-unsatisfiability for every $n > 1$.\footnote{This fact has been exploited in the development of a paraconsistent inference relation which is derivable from the closure conditions on the set of $\Box$ formulae true at a point in a model (Jennings and Schotch, 1980) (Schotch and Jennings, 1989).} Indeed, it follows from our definition of a graph, particularly the fact that graphs contain no self-loops, that for every graph $G$ there is an instance $I_G = (F, c)$ of \textit{CHRMM NMBR} such that for some $r > 1, \exists f \in F : c(f) \leq r$. But as any such $f$ is easily transformed to a satisfying $(r + 1)$-ary model $\mathfrak{M}$ where $|\mathfrak{M}|$ is bounded above by $r + 1$ (and conversely), it follows that the complexity of determining membership in $\text{COLR}_{K_n}$ ($n \geq 1$) is (polynomially) equivalent to that of determining membership in $F$, where $r$ is restricted to $n$. In sum, we therefore have:

\textbf{Theorem 2.3.} Each logic $K_n$ has associated with it a distinct decision problem, corresponding to membership in $\text{COLR}_{K_n}$ for each $n \in \mathbb{Z}^+$, where in particular:

\begin{align*}
\text{COLR}_K &= \text{COLR}_{K_1} = \emptyset. \\
\text{COLR}_{K_2} &\in P. \\
\forall n > 2, \text{COLR}_{K_n} \text{ is NP-complete.}
\end{align*}

\section{A Modal Formulation of 4CT}

\subsection{Motivations}

The history of the Four Colour Theorem (Every planar graph is 4-colourable)\footnote{Contrapositively, if a graph $G$ is 4-uncolourable, then $G$ cannot be embedded in the plane without edge crossings.} (4CT) provides somewhat of a unique, computationally oriented, impetus for exploring logical representations of this, and other, propositions of graph theory. While it is generally acknowledged today to be true, the relatively short history of 4CT as a problem has been fraught with false promises in the form of fallacious ‘proofs’ (see (Thomas, 1998) for a survey). Although Appel and Haken (A&H) produced a proof in 1976 (Appel and Haken, 1977), (Appel, Haken, and Koch, 1977), that was greatly simplified by Roberston et al. in the late 1990s (Roberston et al., 1996), its validity has not been accepted without controversy, essentially because, as the authors of the more recent proof note, “(i) part of the … proof uses a computer, and cannot be verified by hand, and (ii) even the part of the proof that is supposed to be checked by hand is extraordinarily complicated and tedious, and as far as we know, no one has made a complete independent check of it” (Roberston et al., 1996). To remedy (ii) Robertson et al simplify matters insofar as they produce a version of that particular part of the proof (viz., “unavoidability”) which “can be checked by hand in a few months, or, using a computer, … can be verified in a few minutes” (Roberston et al., 1996). However, with respect to (i), their proof remains as controversial as A&H’s.

But if 4CT is representable as a sentence of $K_n$, or any decidable system,\footnote{Decidability for $K_n$ can be proved by applying the filtration method discussed in (Chellas, 1980).} this would imply the existence of an effective method for demonstrating its truth (or falsity) – small comfort, of course, since it wouldn’t follow that any such method is feasibly discoverable or usable. Moreover, one
might wish to argue that the latest proof of 4CT counts more than amply with respect to answering
questions which normally motivate the desire to show decidability, especially as it has furnished us
with a quadratic time algorithm for 4-colouring planar graphs (Roberston et al., 1996). Nevertheless,
the question remains as to whether there is a non-constructive proof for the existence of an effective
method, and also one that can be adapted to provide decidability results for other propositions
of graph-theory. Given the tight connection between \( K_n \) and colouring problems, culminating in
Theorem 2.3, a natural extension of the Aggregative Programme in modal logic therefore consists of
an exploration of the extent to which 4CT, among other graph-related statements, is representable
in \( K_n \).

We initiate this exploration below. Our new contribution to the literature on 4CT is to show how
to formulate a modal rule schema, \([4ct]\) which universally preserves 5-validity iff 4CT is true. In this
way we also show that \( K_n \) can be used to represent an expressivity requirement for any language
functionally complete with respect to the graphic representation of optimization problems. To il-
lustrate, suppose that \( L \), a language for the graphic representation of optimization problems, does
not distinguish between edge-crossings and nodes. For example, suppose that \( L \) represents nodes
of a graph \( G \) by line-crossings in a planar embedding of \( G \). Then, if the schema \([4ct]\) universally
preserves 5-validity, \( L \) cannot unambiguously represent any optimization problem \( P \) with graphic
representation \( G \), for any \( G \) whose uncolouring formulation is 5-valid.

![Figure 3.1.1: The graph on the bottom is the result of contracting the edge \((x, y)\) in the graph on
the top. That is, \((x, y)\) is replaced by the node \( w \), where \( w \) is joined by an edge to every node to
which either \( x \) or \( y \) is joined. Multiple edges and loops are deleted.](image)

### 3.2 Forbidden Minors

A graph \( H \) is a forbidden minor of a graph \( G \) iff \( H \) is isomorphic to either \( K_5 \) or \( K_{3,3} \), and a
graph \( H \) is a minor of a graph \( G \), written \( G \succ H \), if \( H \) is a subgraph of a graph obtainable from
\( G \) by a finite sequence of edge contractions (see Figure 3.1.1). Moreover, it’s not hard to see that if \( H \)
has \( k \) nodes then \( G \succ H \) iff \( G \) has \( k \) vertex-disjoint connected subgraphs \( G_1, G_2, \ldots, G_k \) such that for
each edge \((v_i, v_j)\) in \( H \), there is an edge \((v_q, v_r)\) of \( G \) with \( v_q \in VG_i \) and \( v_r \in VG_j \) (Bollobás, 1991).

Using this fact, our strategy is to formulate for every connected graph \( G \), a non-trivial sentence
schema \( K(G)(= K_5(G) \lor K_{3,3}(G)) \) such that \( K(G) \) is 5-valid iff \( G \) has a forbidden minor.\(^{12}\)

The phrase “non-trivial sentence schema” here is intended to convey two important facts. First,
we give a schema in that for every pair of distinct graphs \( G \) and \( G' \), \( K(G) \) and \( K(G') \) differ at
most in the encoding the mapping \( K \) imparts to their respective adjacency relations \( EG \) and \( EG' \).
Second, it is non-trivial in that, given Wagner’s\(^{13}\) characterization of planarity (Wagner, 1937):

\[
\forall G, G \text{ is planar iff } G \text{ has no forbidden minor,}
\]

\(^{11}\)\(K_{3,3}\) is the complete bipartite graph on node sets of size 3.

\(^{12}\)In what follows we restrict ourselves to connected graphs. For any graph \( G \), if \( H \) is a connected graph then \( G \succ H \)
implies that there is a connected subgraph \( G' \) of \( G \) such that \( G' \succ H \). Thus, since both \( K_5 \) and \( K_{3,3} \) are connected,
our sentence schema applies at least to some subgraph of any graph having one of these as a minor.

\(^{13}\)And independently, Harary and Tutte’s (Harary and Tutte, 1965).
For the sake of generality, we formulate a schema $K_m(G)$ such that $\forall G, m > 1, G \succ \mathbb{K}_m \Leftrightarrow K_m(G)$ is $m$-valid.

If $G$ is a graph, with $m \in \mathbb{Z}^+$, then $\mathbb{G}^m = \{\pi_1, \pi_2, \ldots, \pi_q, \ldots, \pi_r\}$ is the set of all $m$-partitions $\pi_q = \{G^q_1, \ldots, G^q_m\}$ of $G$ into pairwise vertex-disjoint connected subgraphs of $G$. And, if $G$ is a graph with a minor $M$ such that $V(M) = \pi_q \in \mathbb{G}^m$, then $G[\pi_q]$ is the set of pairs $(u^q_i, v^q_j)$ ($i \neq j$) satisfying:

\begin{align*}
  s &\in [V G^q_1], \quad t \in [V G^q_2], \\
  u &\in V G^q_1, \quad v \in V G^q_2, \quad \text{and} \\
  \exists e = (u, v) \in E G.
\end{align*}

In other words, $G[\pi_q]$ is the set of $G$-edges, none of which occurs in any one subgraph in $\pi_q$, and each of whose nodes is appropriately labelled. Then if $G$ is a graph, $m \leq |V G|$, and $\pi_q = \{G^q_1, \ldots, G^q_3\}$ is in $\mathbb{G}^m$, the schemas $M^\pi_q(G)$, and $K_m(G)$ are defined:

\begin{align*}
  M^\pi_q(G) &:= \bigcap_{i=1}^m \bigcap_{s=1}^{\text{dim}(G^q_i)} u^q_i \rightarrow \bigwedge \Box F(G[\pi_q]). \\
  K_m(G) &:= \bigvee_{q=1}^r M^\pi_q(G).
\end{align*}

3.2.2 The schema $K_{3.3}(G)$

By forcing models to ignore 4-colourings of $\mathbb{K}_{3.3}$, the aggregative tendencies of formulae on quinary frames can be exploited to adapt the strategy used in formulating $K_m(G)$ to the present case.

If $G$ is a graph, then let $\mathbb{G}^{3,3}$ be the set $\{\varpi_1, \varpi_2, \ldots, \varpi_q, \ldots, \varpi_r\}$ of all pairs $\varpi_q = \{D^q_1, D^q_2\}$ that satisfy the following conditions:

\begin{align*}
  \exists G_1, G_2 \subseteq G \text{ such that } D^q_1 \in \mathbb{G}^3_1, \quad D^q_2 \in \mathbb{G}^3_2, \quad \text{and} \\
  V G_1 \cap V G_2 = \emptyset, \quad \text{and} \quad V G_1 \cup V G_2 = V G.
\end{align*}

In other words, $\varpi_q$ is a partition of $G$ into two distinct 3-partitions, $D^q_1 = \{G^q_1, G^q_2, G^q_3\}$ and $D^q_2 = \{G^q_4, G^q_5, G^q_6\}$, of vertex-disjoint subgraphs $G_1$ and $G_2$ of $G$, respectively. Evidently then, a graph $G$ has the forbidden minor $\mathbb{K}_{3.3}$ if and only if such a partition exists, with edges between the cells of the partition corresponding to the relevant edges of $\mathbb{K}_{3.3}$. That is, we have:

**Proposition 3.1.** \(\forall G, G \succ \mathbb{K}_{3.3}\) iff there exists some $\varpi_q$ such that $\forall i, j \in [3], \exists e = (s, t) \in E G$ with $s \in V G^q_i$ and $t \in V G^q_j$.

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14 The reader familiar with **Hadwiger’s Conjecture** ($\forall G, n \in \mathbb{Z}^+$, if $G$ is $n$-uncolourable then $G \succ K_{n+1}$) will perceive that we can prove as a corollary that Hadwiger’s Conjecture is true iff for any $G$, the rule schema $\frac{\text{UNCOL}(G)}{K_{n+1}(G)}$ preserves $(n+1)$-validity.

15 A similar strategy is used in (Nicholson, 2001) to develop a construction with applications towards a possible proof of Hadwiger’s Conjecture.
So, to represent any edge \((s,t)\) in a potential \(K_{3,3}\) minor of \(G\) (relative to \(\varpi_q\)), we define \(G(D^q_1,D^q_2)\) (see Figure 3.2.1) as the set of pairs \((s^q_{i,j},t^q_{k,l})\) satisfying:

\[
\begin{align*}
&i, k \in [3], \\
&j \in [V_G^{q_1}], l \in [V_G^{q_2}], \\
&s \in V_G^{q_1}, t \in V_G^{q_2}, \text{ and} \\
&e = (s,t) \in E_G.
\end{align*}
\]

(3.10)

Then the schemas \(M^{(D^q_1,D^q_2)}(G)\) and \(K_{3,3}(G)\) are defined as:

\[
M^{(D^q_1,D^q_2)}(G) := \bigvee_{q=1}^r M^{(D^q_1,D^q_2)}(G).
\]

(3.15)

3.2.3 The schema \(K(G)\)

In the interest of satisfying restrictions imposed on the length of this paper, we establish our central result pertaining to 4CT, viz.,

**Theorem 3.2.** \(\forall G, \overline{\text{UNCOL}(G)}_{K(G)}\) preserves 5-validity iff 4CT is true.

For purposes of proof, we sketch the ‘hard’ direction of the following lemma:

**Lemma 3.3.** \(\forall G, K(G)\) is 5-valid iff \(G\) is not planar,

where for any graph \(G\), the schema \(K(G)\) is defined as:

\[
K(G) := K_{3,3}(G) \cup K_5(G).
\]

(3.16)

**Proof.** Let \(G\) be a planar graph. Then by Wagner’s theorem (3.1), \(G \not\cong K_{3,3}\) and \(G \not\cong K_5\). Thus, from Proposition 3.1 it follows that \(\forall \varpi_q = (D^q_1,D^q_2) \in G^{3,3}, \exists i, j \in [3] \text{ such that no } e \in E_G \text{ has endpoints in both } G_i^{q_1} \text{ and } G_j^{q_2}\). Consequently there is a 4-partition \(\pi'_q = \{c^q_1,\ldots,c^q_4\}\) satisfying:

\[
\begin{align*}
\forall h \in [3], \exists c^q \in \pi'_q: V_G^{q_1} &\subseteq c^q, \\
\forall h \in [3], \exists c^q \in \pi'_q: V_G^{q_2} &\subseteq c^q, \\
\exists c^q, i, j \in [3]: V_G^{q_1} &\cup V_G^{q_2} \subseteq c^q, \text{ and} \\
\forall e \in G(D^q_1,D^q_2), \forall c^q \in \pi'_q e \not\subseteq c^q.
\end{align*}
\]

(3.17) (3.18) (3.19) (3.20)
We use \( \pi' = \{c_1', \ldots, c_n'\} \) to define a model \( \mathcal{M} = (\mathcal{F}, V) \), on a quinary frame \( \mathcal{F} = (U, R) \) such that for some \( x \in U, x \notin V(K_{3,3}(G)) \). To this end let \( \pi' \) be the set \( \{c_i' \mid i \in [4] \text{ and } c_i' = \bigcup_{q=1}^4 c_i^q \} \). Then where:

\[
U := \pi'(\{c_1', \ldots, c_i'\}) \cup \{x\}, \quad \text{for some new variable } x, \tag{3.21}
\]

\[
R := \{(x, c_1', \ldots, c^*_i)\}, \quad \text{and} \tag{3.22}
\]

\[
\forall p \in \phi, c^*_i \in V(p) \iff p \in \pi^*, \tag{3.23}
\]

it is straightforward to show that \( x \notin V(K_{3,3}(G)) \). By similar reasoning, the model \( \mathcal{M} \) can be augmented in such a way that \( K_5(G) \) is simultaneously false at \( x \). Whence \( K(G) \) is not 5-valid if \( G \) is planar.

\[
\square
\]

4 The complexity of \( K_n \)

Ladner (Ladner, 1977) has shown that the satisfiability question for any modal logic \( S \) such that \( K \leq S \leq S_4 \) is PSPACE-complete.\(^6\) The proof relies upon a reduction from the satisfiability problem for QBFs (Quantified Boolean Formulae), known to be PSPACE-complete (Stockmeyer and Meyer, 1973). For any QBF \( A \) of the form \( Q_1 p_1 Q_2 p_2 \ldots Q_m p_m A' \), where each \( Q_i \) is either \( \forall \) or \( \exists \), and \( A' \) is a quantifier-free formula containing all and only propositional letters \( p_1, \ldots, p_m \), we can derive a modal formula \( B \) such that \( B \) is satisfiable in the modal logic \( S \) iff \( A \) is a satisfiable QBF. To do so, we define \( B \) so that a modal tree-structure satisfying it has leaves that mimic an appropriate set of truth-assignments to the propositional letters in \( A \). Halpern (Halpern, 1995) has shown that the PSPACE bound is robust under various restrictions on the number of propositional letters appearing in modal formula \( B \). At the same time, he shows that bounding the maximum depth of modal formulae, measured in terms of the nesting of modal operators, can in some cases reduce the complexity of modal SAT. In particular, for formulae of depth \( < k \), \( k \in \mathbb{N} \), the satisfiability problem for logics \( K \) and \( T \) is NP-complete; for \( S_4 \), the problem remains PSPACE-complete.\(^7\) We generalize these established results, showing for the first time that they extend to logics \( K_n \).

4.1 \( K_n \)-SAT

Due to the weakly-aggregative nature of logics \( K_n \), \( n \geq 2 \), the Ladner/Halpern proofs of PSPACE bounds on satisfiability do not immediately go through, since these proofs rely upon the power of the \( \Box \) operator to force various formulae to come out true in single worlds along a modal tree-structure. For any logic weaker than \( K = K_1 \), however, the presence of two formulæ \( \Box A \) and \( \Box B \) at some point \( x \) in our model is not generally enough to guarantee that both \( A \) and \( B \) are true at a single point \( y \) in any \( n \)-tuple related to \( x \). However, by suitably finessing the definition of the modal formula corresponding to a given QBF, we are able to generalize the result, proving:

**Theorem 4.1.** For any modal logic \( S \), and any \( n \geq 1 \), if \( K_n \leq S \leq S_4 \), then the satisfiability problem for \( S \) (\( S \)-SAT) is PSPACE-complete.

While the proof of this claim is outside the scope of this paper (see (Allen, 2003)), we present a generalization of Halpern, Theorem 4.1 (Halpern, 1995), giving an NP-completeness result for \( K_n \)-SAT in the presence of bounded operator depth.

The depth of modal formula \( B \), written \( dpt(B) \), is the maximum nesting of its modal operators, defined inductively. For any propositional letter \( p \), \( dpt(p) = 0 \); \( dpt(\neg A) = dpt(A) \); \( dpt(A \land B) = \max(dpt(A), dpt(B)) \); and \( dpt(\Box A) = (1 + dpt(A)) \). This allows a theorem:

**Theorem 4.2.** For any \( k \in \mathbb{N} \), and any logic \( K_n \), the \( K_n \)-satisfiability problem, restricted to \( \{A \mid dpt(A) < k\} \), is NP-complete.

**Proof.** The reduction from an NP-complete problem is immediate, since every propositional formula \( A \) is also a formula of our modal language. Furthermore, \( dpt(A) = 0 \) for any such \( A \), and thus we

---

\(^6\) For a simpler presentation of important elements of the proof the reader may also want to see Halpern & Moses (Halpern and Moses, 1992), and Halpern (Halpern, 1995).

\(^7\) Logic \( K \) is our \( K_1 \), while \( T \) adds axiom \((\Box A \rightarrow A)\) to \( K \), and \( S_4 \) adds \((\Box A \rightarrow \Box \Box A)\).
have that propositional SAT reduces immediately to $K_n$-SAT with depth bounded by any $k \in \mathbb{N}$ and any $K_n$. We need only show that an NP decision procedure exists for bounded-depth $K_n$-SAT.

Let $B$ be a modal formula of length $|B| = m$, with depth less than $k \in \mathbb{N}$. Let $\text{Prop}(B)$ be the set of propositional letters in $B$. Now, $B$ is $K_n$-satisfiable iff some $(n+1)$-ary relational model $\mathfrak{M}$ makes $B$ true. It is easy to see that any such $\mathfrak{M}$ corresponds to a series of assignments, $V_i : \text{Prop}(B) \rightarrow \{1, 0\}$, one for each point in $\mathfrak{M}$. Once such a series is given, evaluating $B$ is straightforward. A nondeterministic machine $N$ needs only to guess some set of assignments, tracking the right relationships between them, and then perform evaluation as usual. To make the presentation clear, we label assignments (although $N$ itself need not use any such device). For logic $K_n$, each label is some string from the alphabet $\{0, 1, \ldots, n-1, *\}$, of the form:

$$w = \{0, 1, \ldots, n-1\} \circ (* \{0, 1, \ldots, n-1\})^r \quad (r \geq 0).$$

That is, each label is a sequence of numbers from 0 to $(n-1)$, each separated by *. Intuitively then, for any $w$, and any $0 \leq x \leq n-1$, assignment $V_{w,x}$ is a successor of assignment $V_w$. The satisfaction conditions are as usual for propositional letters and boolean connectives; for modal operators, the condition is given by:

$$\forall A, V_w \models \Box A \iff (\exists i : 0 \leq i \leq n-1) V_{w*i} \models A.$$  

(It is easy to see that this condition corresponds to the definition of $V(\Box A) [1,1]$.) So, for any modal subformula $B'$ of $B$, occurring with $dpt(B') \leq dpt(B) < k$, machine $N$ will need to look at not more than some constant number ($n^{dpt(B')}$) of assignments in order to determine whether $B'$ is satisfied or not.

To determine the upper limit on the number of assignments $V_w$ our machine $N$ will have to guess for logic $K_n$ and formula $B$, we observe that for every depth $j \leq dpt(B) < k$, we require at most $n$ such new assignments, one for each modal operator occurring at depth $j$. Since there must be not more than $|B| = m$ such operators occurring in the entire formula, we require not more than $(n \cdot m)^{dpt(B)} + 1$ such assignments, each of which assigns not more than $|B| = m$ values to propositional letters. The overall size of the assignment-set that must be guessed is thus not more than $(n^{dpt(B)} \cdot m^{dpt(B)+1}) + 1$, which, since $dpt(B) < k$, is strictly less than $(n^k \cdot m^{k+1}) + 1$. Thus, $N$ needs only guess a series of assignments with overall size polynomial in $m = |B|$.

\section{Conclusion: Open problems}

The weakly-aggregative modal logics open up a number of avenues for further investigation from a complexity-theoretic point of view, particularly given the capacity of such logics to express interesting graph-theoretic and combinatorial properties. We mention but one here. In the context of a study of modal epistemic logics, Vardi uses a version of neighbourhood semantics in order to address concerns about ‘omniscient agents’ (Vardi, 1989). In this context, he extends the PSPACE-completeness results to modal logics outside of the range between $K$ and $S4$, and conjectures that this bound depends upon the presence of the fully-aggregative $[K_1]$ axiom: $(\Box A \land \Box B) \rightarrow \Box (A \land B)$. That is, he conjectures, $S$-SAT is PSPACE-complete iff modal logic $S$ contains said axiom, and is NP-complete otherwise. We have shown that SAT for $K_n$ logics may yet be PSPACE-complete in the presence of axioms strictly weaker than that given, at least in the context of the usual Kripke-style semantics. It is worth investigating how these results translate over into the semantic framework of Vardi, and how they bear on his conjecture.\footnote{We measure length $|B|$ in number of symbols. If what we want rather is to measure $|B|$ as some function, say, of $\log |\tau|$, for $\tau$ an alphabet of symbols, it is easy to translate what follows into those terms.}

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